2. Governing equations - Which equations do we need to solve?

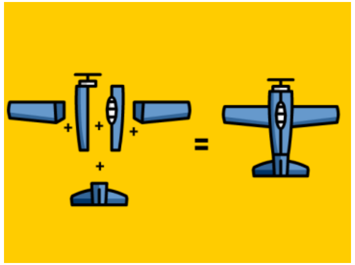
- Transient or steady-state?
- 2D or 3D?
- Incompressible or compressible flow?
- Viscous or non-viscous?
- Single-phase or multiphase?
- What about turbulence?

Conservation Laws

Mass:

$$\left(\frac{dm}{dt}\right)_{syst.} = 0$$

The mass within a material region is constant



Momentum:

$$\left[\frac{d(m\boldsymbol{U})}{dt}\right]_{syst.} = \sum F$$

The time rate of change of the linear momentum of a material region is equal to the sum of the forces acting on the region

Newtons 2nd laws:

$$m \cdot \boldsymbol{a} = \sum F$$

Conservation Laws *Newtons 2nd law*

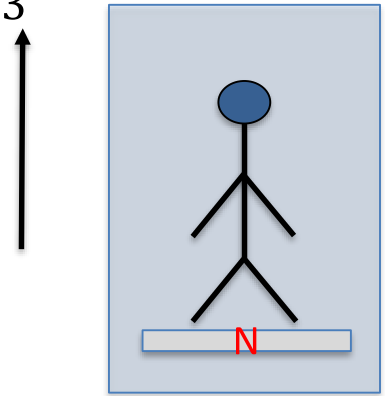
- Example from undergraduate physics, $\sum F = m \cdot \boldsymbol{a}$

Person has a weight of, $W = 75kg$.

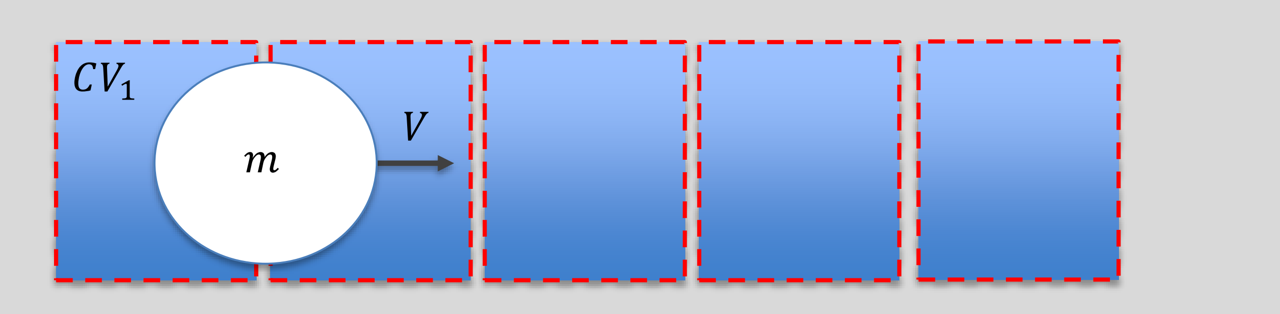
Elevator accelerates upwards $1/3\ g$

What is the weight *N* in the bottom showing in kg?

$$a = \frac{1}{3} \cdot g$$



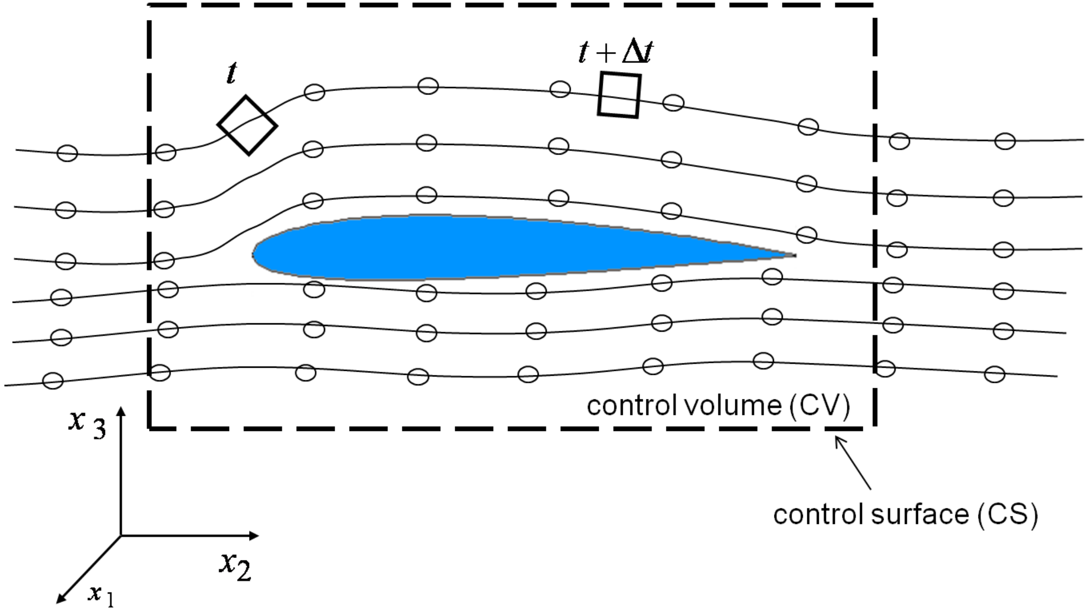
Example – *Euler vs. Lagrangian*



- <u>Lagrange: </u>Follow trajectory of particle with fixed mass
- <u>Euler: </u>Calculate the forces exerted on a CV

Conservation Laws

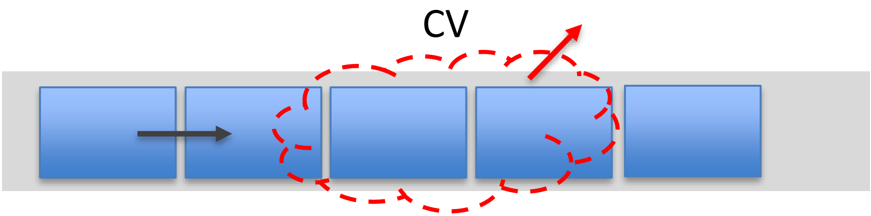
It is not in our interest to follow the trajectory of a fluid particle. We wish to define a non-moving fluid environment.



Conservation Laws

- Mass is conserved:

$$\frac{d}{dt}\underbrace{\left(\int_{CV}\rho\, dV\right)}_{\boxed{\text{Mass inside the CV}}} + \int_{Surf}\underbrace{\rho(\boldsymbol{V_r}\cdot\boldsymbol{n})dA}_{\boxed{\text{Mass flux crossing the boundaries}}} = 0$$



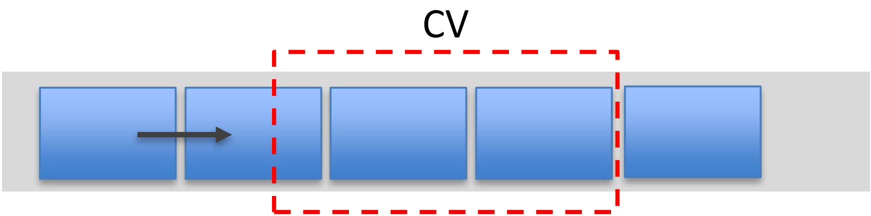
Fluid through a pipe – *variable, moving* *CV*

Conservation Laws

- Mass is conserved:

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) \rightarrow \int_{CV} \frac{\partial(\rho)}{\partial t} \, dV$$

Fixed CV



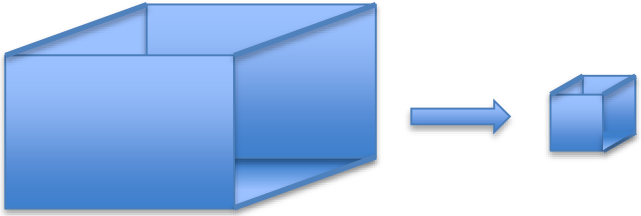
Fluid through a pipe – *fixed, non-moving* *CV*

Conservation Laws – *Mass*

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV + \int_{Surf} \rho(\boldsymbol{V} \cdot \boldsymbol{n}) dA = 0$$

Transform the last integral to a volume integral (Gauss' theorem):

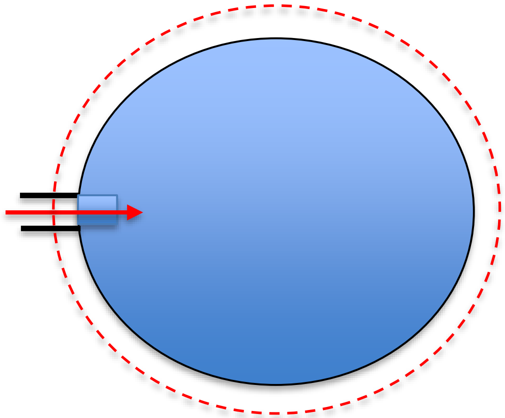
$$\int_{cv} \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i}\right] dV = 0$$



By letting $dV \rightarrow 0$ we get the continuity equation in differential (local) form

Example Mass conservation

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{Surf} \rho(\boldsymbol{V_r} \cdot \boldsymbol{n}) dA = 0$$



- One-dimensional inlet:

$$\int_{Surf} \rho(\boldsymbol{V_r} \cdot \boldsymbol{n}) dA = \dot{m}_{in} \cdot V_{in} = (\rho A V)_{in} \cdot V_{in}$$

Momentum on integral form

$$\left(\frac{d(m\boldsymbol{U})}{dt}\right)_{syst} = \sum F = \frac{d}{dt}\left(\int_{CV} \boldsymbol{V}\,\rho\,dV\right) + \int_{CS} \boldsymbol{V}\,\rho(\boldsymbol{V}\cdot\boldsymbol{n})\,dA$$

Consider surface forces ***S*** and body forces, ***B***

Using Gauss' theorem and assume constant CV volume:

$$\int_{cv}\left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_j u_i)}{\partial x_j}\right] dV = \int_{cv} \rho B_i\,dV + \int_{cs} S_i\,dA$$

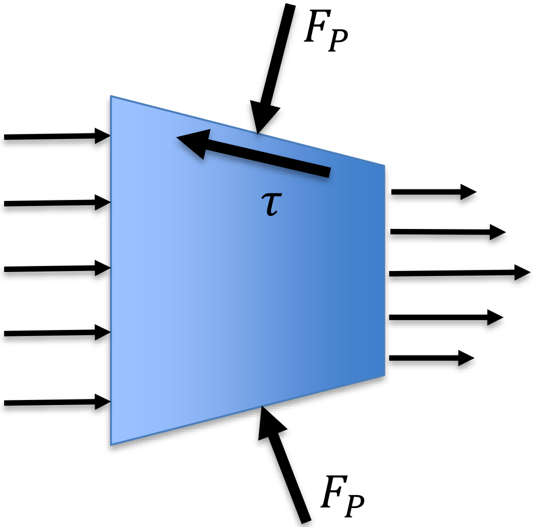
Conservation Laws – one-dim. simple form

- Uniform velocity and density over the inlet / exit:

$$\sum F = \frac{d}{dt}\left(\int_{CV} \boldsymbol{V}\,\rho\, dV\right) + \sum(\dot{m}\boldsymbol{V}_i)_{out} - \sum(\dot{m}\boldsymbol{V}_i)_{in}$$

- Steady, one inlet and outlet:

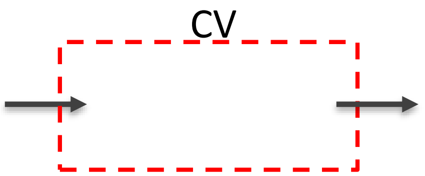
$$\sum \boldsymbol{F} = \dot{m}(\boldsymbol{V}_{in} - \boldsymbol{V}_{out})$$



Conservation Laws – one-dim. simple form

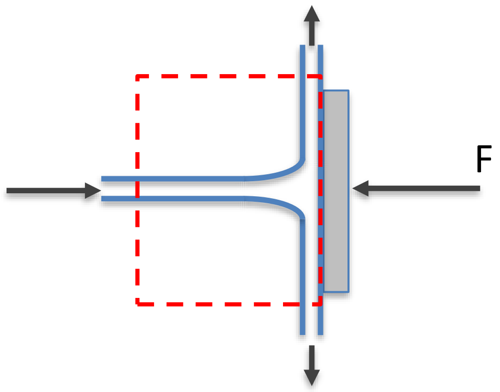
Mass:

$$\left(\frac{dm}{dt}\right)_{cv} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

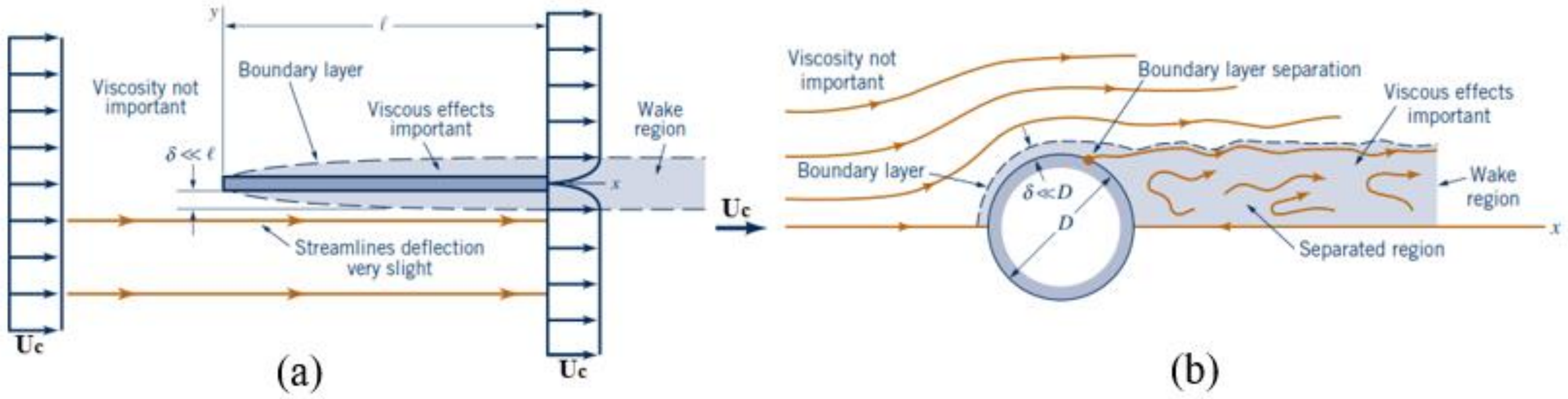


Momentum:

$$\sum F = \sum (\dot{m}u)_{in} - \sum (\dot{m}u)_{out}$$

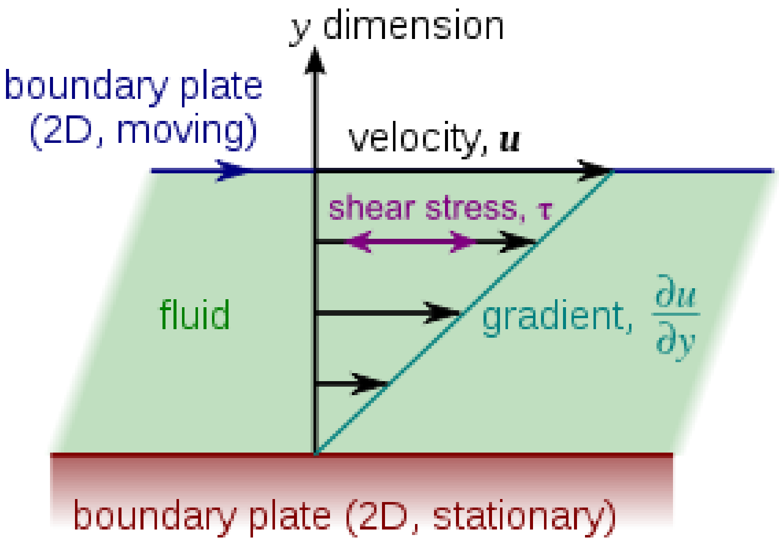


Discuss the difference of forces:



Which forces are important in both cases?

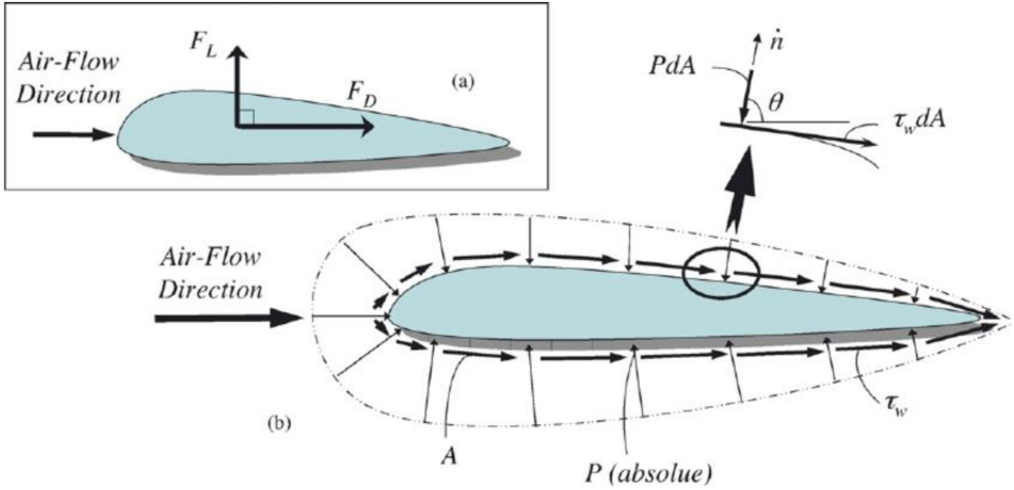
Forces *– Shear force*



Forces *– which forces act in a fluid?*

Surface forces are *pressure* and *viscous stress*
Body forces are gravity, and other external forces

Stress is divided into pressure p (1x3) and viscous stress τ_{ij} (3x3):

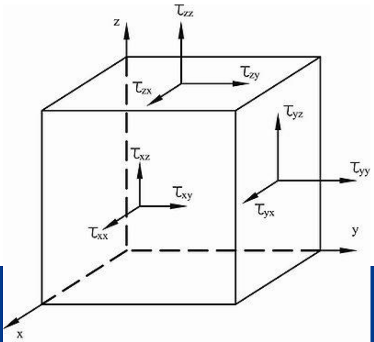


NTNU

Momentum on differential form (2 of 3 equations)

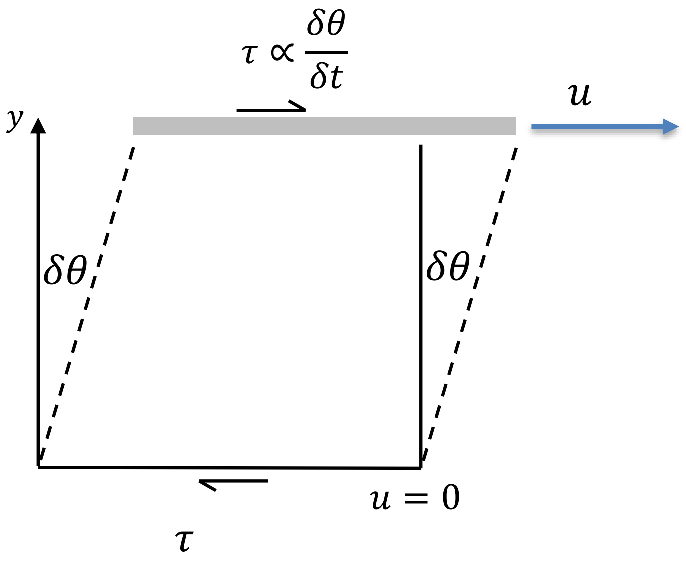
$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}+\frac{\partial(\rho wu)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\rho f_x$$

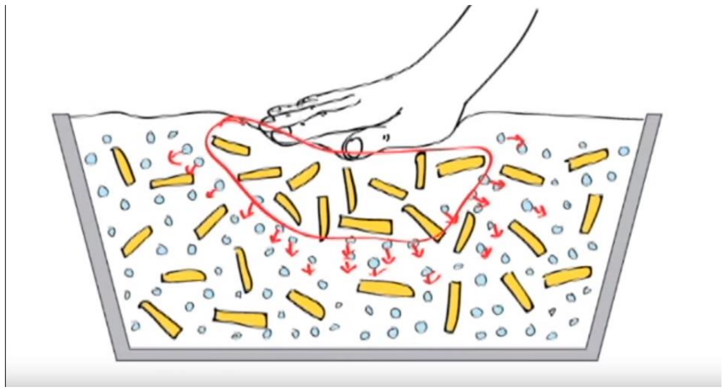
$$\underbrace{\frac{\partial(\rho v)}{\partial t}}_{\text{Temporal}}+\underbrace{\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho vv)}{\partial y}+\frac{\partial(\rho wv)}{\partial z}}_{\text{Convection}}=\underbrace{-\frac{\partial p}{\partial y}}_{\text{Pressure}}+\underbrace{\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}}_{\text{Viscous}}+\underbrace{\rho f_y}_{\text{Other Forces}}$$



Navier-Stokes equations

- Newtonian fluid (simple molecules, water, oil, air) → Property of the fluid
- Non-Newtonian (salt solutions, blood, ketchup, starch suspensions)



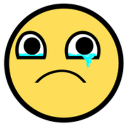


Strain rate $\frac{\delta\theta}{\delta t}$ proportional to viscous stress. $\tau = \mu \frac{du}{dy}$

Navier-Stokes equations (Full version, 1 of 3 equations)

$$\frac{\partial u}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}+\frac{\partial(\rho wu)}{\partial z}$$
$$=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(-\frac{2}{3}\mu\nabla\cdot\boldsymbol{U}+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x$$

Local form !



Navier-Stokes equations (incompressible version)

- Incompressible flow means $\nabla \cdot \boldsymbol{U} = \mathbf{0} \rightarrow \frac{\boldsymbol{d\rho}}{\boldsymbol{dt}} = \mathbf{0}$
- Constant density and viscosity:

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f_x$$

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial(uv)}{\partial x} + \frac{\partial(vv)}{\partial y} + \frac{\partial(wv)}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho f_y$$

Navier-Stokes equations simplifications

- Steady-state

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+\frac{\partial(uu)}{\partial x}+\frac{\partial(vu)}{\partial y}+\frac{\partial(wu)}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho f_x$$

Euler equations simplifications

- Steady-state
- Non-viscous

$$\rho\left(\frac{\partial(uu)}{\partial x}+\frac{\partial(vu)}{\partial y}+\frac{\partial(wu)}{\partial z}\right)=-\frac{\partial p}{\partial x}+\cancel{\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}+\rho f_x$$

$$\rho\left(\frac{\partial(uu)}{\partial x}+\frac{\partial(vu)}{\partial y}+\frac{\partial(wu)}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho f_x$$

Can be written: $\boxed{\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)}$

Euler equations simplifications

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + f_y$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z$$

Bernoulli simplifications

It can be shown that by assuming *irrotational flow* / integration along a *streamline*

$$d\left(\frac{1}{2}V^2\right) + \frac{dp}{\rho} + g \cdot dz = 0$$

Integration gives:

$$\Delta p + \frac{1}{2}\rho(V_2^2 - V_1^2) + \rho g(z_2 - z_1) = 0$$

Bernoulli example

A gravity tank with water is emptied. Estimate the force necessary to hold the plate.

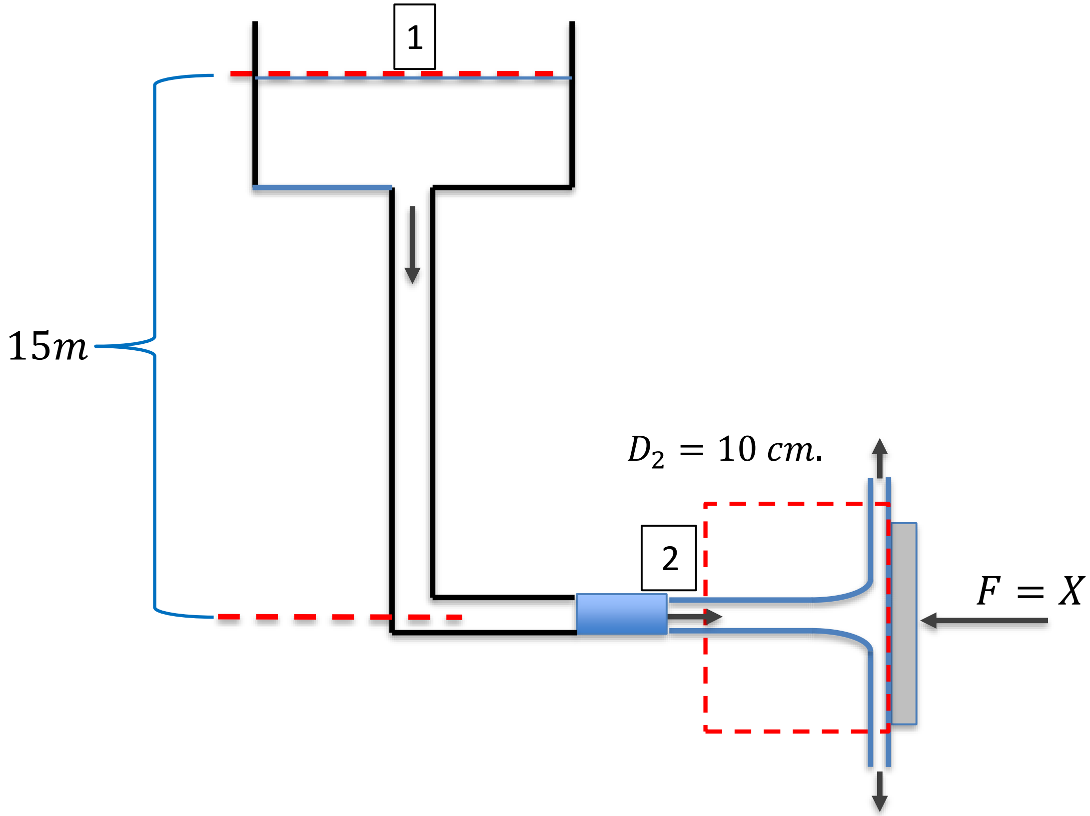
Assume $V_1 = 0$ and the static pressure is atmospheric at 1 and 2.

What is a necessary condition for the above assumption to be valid?

Neglect friction.

Diameter of the outlet pipe is 10 cm.

What if the plate is inclined by an angle of 5 degrees?

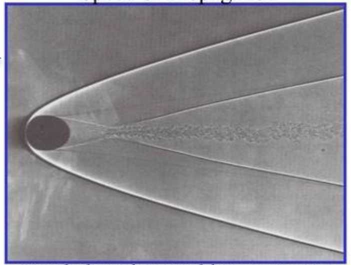


Classification of flows *- examples*

- Internal / External Flow
- Stokes flow (creeping flow): $Re \ll 1$

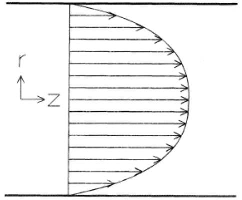
- Incompressible / compressible

Speed of Propagation

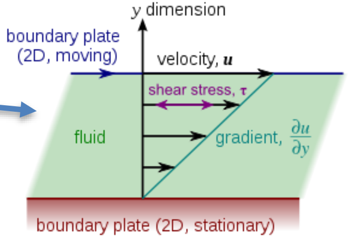


"speed of sound is one of the most important quantities in the study of incompressible flow"
- Anderson : *Moderns Compressible Flow*

- Poiseuille flow



- Couette flow



Classification of flows *- examples*